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ON STABILITY WHEN A SINGULAR POINT IS NON-ISOLATED*

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Assuming that m zero roots correspond to the non-isolated singular point of the equations of disturbed motion, the problem of stability when some of the remaining roots are pure imaginary, while others have a negative real part in inessentially singular cases, is considered. This situation is typical for non-holonomic systems. The stability and instability theorems obtained are used to study the stability of the rotations of a celtic stone on the boundary of the domain of stability.

1. Let the system of equations of the disturbed motion with holomorphic right-hand sides and explicitly isolated linear approximation be

$$x^{\cdot} = X(x, y), \quad y^{\cdot} = Ay + Y(x, y) \tag{1.1}$$

where x is an m-vector, y is an $(N-m)$-vector, and A is a constant $(N-m) \times (N-m)$ matrix. We assume that

$$X(x, 0) = Y(x, 0) \equiv 0 \tag{1.2}$$

for all x in the neighbourhood of the point $x = 0$. The singular point $x = 0, y = 0$ of system (1.1) is then non-isolated: by (1.2), Eqs.(1.1) admit of the manifold of equilibrium states

$$x = c, \quad y = 0 \quad (c = (c_1, \ldots, c_m), \quad c_k = \text{const}, \quad k = 1, \ldots, m)$$

Under conditions (1.2), we can regard (1.1) as the equations of the disturbed motion of a non-holonomic system with m non-holonomic constraints /1, 2/. It was proposed in /3/ to regard as critical cases only those when the number of zero roots of the characteristic equation is greater than the number of equations of the non-holonomic constraints, while in /4/ it was proposed to call these cases singular.

Definition. We have the K-singular case if conditions (1.2) hold for Eqs.(1.1) and at least one eigenvalue of the matrix A has a zero real part.

2. We consider the K-singular case when the matrix A in Eqs.(1.1) has n pairs of distinct pure imaginary roots and q roots with negative real parts ($N = m + 2n + q$). We rewrite Eqs. (1.1) as

$$\xi^{\cdot} = F_0 + F_1, \quad \eta^{\cdot} = \Lambda\eta + \Phi_0 + \Phi_1, \quad \bar{\eta}^{\cdot} = -\Lambda\bar{\eta} + \bar{\Phi}_0 + \bar{\Phi}_1$$
$$z^{\cdot} = Pz + Z_0 + Z_1, \quad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \tag{2.1}$$

Here, ξ and z are real m- and q-vectors, η, $\bar{\eta}$ are conjugate complex n-vectors, $\bar{\Phi}_0$, $\bar{\Phi}_1$ are complex vector functions, conjugate respectively to Φ_0 and Φ_1, Λ is a matrix of pure imaginary eigenvalues, and the constant $(q \times q)$ matrix P has eigenvalues with negative real parts; the functions with unit subscript are zero when $z = 0$. The arguments of functions with zero subscript are ξ, η and $\bar{\eta}$, and with unit subscript, are ξ, η, $\bar{\eta}$ and z.

In the inessentially singular case we have none of the identities $F_0 = \Phi_0 = \bar{\Phi}_0 = Z_0 \equiv 0$ /5/. We write the functions with zero subscript as the sum of two terms:

$$F_0 = F_0^{(1)}(\xi) + F_0^{(2)}(\xi, \eta, \bar{\eta}), \quad F_0^{(2)}(\xi, 0, 0) = 0; \ldots$$

(we write similar equations for Φ_0, $\bar{\Phi}_0$, Z_0).

We rewrite conditions (1.2) as

$$F_0^{(1)} = \Phi_0^{(1)} = \bar{\Phi}_0^{(1)} = Z_0^{(1)} \equiv 0 \tag{2.2}$$

We can always arrange for the functions $F_0^{(2)}$ and F_1 here to contain no terms that are linear in $\eta, \bar{\eta}$ and z /6/. In the inessentially singular case, the stability problem for the zero solution of system (2.1) can be reduced to the stability problem /5, 6/ for the zero solution of the system (we retain the previous notation)

$$\xi^{\cdot} = F_0, \quad \eta^{\cdot} = \Lambda\eta + \Phi_0, \quad \bar{\eta}^{\cdot} = -\Lambda\bar{\eta} + \bar{\Phi}_0 \tag{2.3}$$

in the critical case of m zero roots in m groups of solutions and n pairs of pure imaginary roots, if the stability problem is solved for terms of finite order in the variables ξ, η, $\bar{\eta}$,

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and we have the identities $F_0^{(1)} = \Phi_0^{(1)} = \overline{\Phi}_0^{(1)} \equiv 0$ for system (2.3).

We reduce system (2.3) to the normal form /7, 8/ up to and including terms of the M-th order. The K-singularity is then preserved. Hence, using the previous notation, we obtain

$$\xi_k^{\cdot} = \sum_{s=1}^{n} a_{ks}\eta_s\overline{\eta}_s + \sum_{l=3}^{M} F_k^{(l)} + F_k \tag{2.4}$$

$$\eta_s^{\cdot} = \lambda_s\eta_s + \sum_{\nu=1}^{m} A_{\nu s}\xi_\nu\eta_s + \sum_{l=2}^{M} \Phi_s^{(l)} + \Phi_s$$

$$\overline{\eta}_s^{\cdot} = -\lambda_s\overline{\eta}_s + \sum_{\nu=1}^{m} \bar{A}_{\nu s}\xi_\nu\overline{\eta}_s + \sum_{l=2}^{M} \overline{\Phi}_s^{(l)} + \overline{\Phi}_s$$

$$0 \equiv F_{k0}^{(l)} = F_{k0} = \Phi_{s0}^{(l)} = \Phi_{s0} = \overline{\Phi}_{s0}^{(l)} = \overline{\Phi}_{s0}\ (F_{k0}^{(l)} = F_k^{(l)}(\xi, 0, 0), \ldots)$$

Here, a_{ks} and $A_{\nu s}$, $\bar{A}_{\nu s}$ are respectively real and complex conjugate coefficients; $F_k^{(l)}$, $\Phi_s^{(l)}$, $\overline{\Phi}_s^{(l)}$ are respectively real and complex conjugate resonance /7, 8/ forms of order l, dependent on $\xi, \eta, \overline{\eta}$; F_k, Φ_s, $\overline{\Phi}_s$ is a set of orders of smallness which are higher than M and depend on $\xi, \eta, \overline{\eta}$. Throughout, $k = 1, \ldots, m$; $s = 1, \ldots, n$.

After the change of variables

$$\eta_s = \rho_s \exp(\lambda_s t + i\theta_s), \quad \overline{\eta}_s = \rho_s \exp(-\lambda_s t - i\theta_s)$$

system (2.4) takes the form /7, 8/

$$\xi_k^{\cdot} = \sum_{s=1}^{n} a_{ks}\rho_s^2 + \sum_{l=3}^{M} H_k^{(l)} + H_k \tag{2.5}$$

$$\rho_s^{\cdot} = \sum_{\nu=1}^{m} b_{\nu s}\xi_\nu\rho_s + \sum_{l=2}^{M} P_s^{(l)} + P_s$$

$$\rho_s\theta_s^{\cdot} = \sum_{\nu=1}^{m} c_{\nu s}\xi_\nu\rho_s + \sum_{l=2}^{M} Q_s^{(l)} + Q_s$$

$$b_{\nu s} = \operatorname{Re} A_{\nu s}, \quad c_{\nu s} = \operatorname{Im} A_{\nu s}, \quad \rho = (\rho_1, \ldots, \rho_n), \quad \theta = (\theta_1, \ldots, \theta_n)$$

Here, $H_k^{(l)}$, $P_s^{(l)}$, $Q_s^{(l)}$ are the respective resonanance forms of system (2.4) of order l in ξ, ρ with coefficients, periodic in θ; the functions H_k, P_s and Q_s are forms in ξ, ρ, which are periodic in t and θ, and have an order higher than M.

We shall assume that $m = 2$ (the generalization to the case $m > 2$ is obvious). We transform in system (2.5) to $(n + 2)$-dimensional spherical coordinates $r, \varphi_1, \ldots, \varphi_{n+1}$

$$\xi_1 = r\cos\varphi_1, \quad \xi_2 = r\cos\varphi_2 \sin\varphi_1$$

$$\rho_\nu = r\cos\varphi_{\nu+2} \prod_{j=1}^{\nu+1} \sin\varphi_j, \quad \rho_n = r\prod_{j=1}^{n+1} \sin\varphi_j$$

$$0 \leqslant \varphi_1, \ \varphi_2 \leqslant \pi, \ 0 \leqslant \varphi_j \leqslant \pi/2 \quad (j = 3, \ldots, n+1),$$

$$\nu = 1, \ldots, n-1$$

and write the corresponding equations only for $r^{\cdot}$ and $\varphi_1^{\cdot}$, taking $M = 2$

$$r^{\cdot} = \Delta^2 [R^{(0)}(\varphi, \theta) + R(r, \varphi, \theta, t)] \tag{2.6}$$

$$\varphi_1^{\cdot} = \Delta \sin\varphi_2 [G^{(0)}(\varphi, \theta) + G(r, \varphi, \theta, t)]$$

$$\Delta = r\sin\varphi_1 \sin\varphi_2, \quad R^{(0)}(\varphi, \theta) = K(\varphi', \theta)\Delta r^{-1} +$$

$$\sum_{s=1}^{n} [(a_{1s} + b_{1s})\cos\varphi_1 + (a_{2s} + b_{2s})\cos\varphi_2 \sin\varphi_1]\rho_s^2\Delta^{-2}$$

$$G^{(0)}(\varphi, \theta) = \cos\varphi_1 K(\varphi', \theta)\Delta r^{-1} + L(\varphi)$$

$$L(\varphi) = \sum_{s=1}^{n} [b_{1s}\cos^2\varphi_1 + \cos\varphi_2 (a_{2s} + b_{2s})\cos\varphi_1 \sin\varphi_1 -$$

$$a_{1s}\sin^2\varphi_1]\rho_s^2\Delta^{-2}, \quad K(\varphi', \theta) = \sum_{s=1}^{n} K_s^{(3)}(\varphi', \theta)\rho_s\Delta^{-1}$$

$$\varphi' = (\varphi_3, \ldots, \varphi_{n+1}), \quad K_s^{(3)}(\varphi', \theta) = P_s^{(3)}(\xi, \rho, \theta)\Delta^{-3}$$

(since the least order of inner resonances in this case is three, the functions $P_s^{(3)}(\xi, \rho, \theta)$ are independent of ξ).

We write Δ^2 and $\Delta\sin\varphi_2$ on the right-hand sides of Eqs.(2.6) as common factors for the entire right-hand side of these equations, inasmuch as we have the K-singular case and there are no terms linear in $\eta, \overline{\eta}, z$ in the equations for $\xi^{\cdot}$ in system (2.1). The functions $\rho_s^2\Delta^{-2}$

can be written as

$$\cos^2\varphi_3,\quad \cos^2\varphi_4\sin\varphi_3,\ \ldots,\ \cos^2\varphi_{n+1}\prod_{\nu=1}^{n}\sin^2\varphi_\nu,\quad \prod_{\nu=1}^{n+1}\sin^2\varphi_\nu$$

Thus these functions do not vanish simultaneously for any possible values of $\varphi_3, \varphi_4, \ldots, \varphi_{n+1}$.

Theorem 1. Assume that 1) for all s the coefficients b_{1s} are of the same sign and that we have the inequalities $(a_{2s}+b_{2s})^2+4a_{1s}b_{1s}<0$; 2) $\min_\varphi |L(\varphi)| > \max_{\varphi',\theta} |K(\varphi',\theta)/2|$.

Then, the zero solution of system (2.1) is Lyapunov-stable.

Proof. Under condition 1) (condition 2)) the functions $L(\varphi)$ $(G^{(0)}(\varphi,\theta))$ become of fixed sign. We consider the function $V=r\exp(-h\cos\varphi_1)$, where h is a real number. We calculate the derivative with respect to t of this function, using the system of equations of the disturbed motion (2.6)

$$V^{\cdot}=V\Delta^2r^{-1}(hG^{(0)}(\varphi,\theta)+R^{(0)}(\varphi,\theta)+hG(r,\varphi,\theta,t)+R(r,\varphi,\theta,t))$$

For sufficiently small r, we can choose h in such a way that the derivative $V^{\cdot}$ is of fixed sign, while $VV^{\cdot}\leqslant 0$; in this case, V satisfies all the conditions of Rumyantsev's theorem /9/ on stability with respect to some of the variables.

Note that V is chosen in the same way in /10/.

Notes. 1°. If the coefficients b_{2s} are of the same sign of all s, and we have $(a_{1s}+b_{1s})^2+4a_{2s}b_{2s}<0$, then, under the corresponding condition 2) of Theorem 1, the zero solution of system (2.1) is likewise Lyapunov-stable.

2°. If $m=1$ (i.e., there is no zero root), $n\geqslant 1$, and there are no inner resonances (in this case $K(\varphi',\theta)\equiv 0$), then Theorem 1 is the same as the theorem of /10/ (it suffices to put $a_{2s}=b_{2s}=0$ in the conditions of Theorem 1).

3°. In the case $m>2$, $n\geqslant 1$, and there are no inner resonances, we can rewrite Theorem 1 in the light of Note 1° as follows.

Theorem 2. If there exists for all s at least one k such that all the b_{ks} are of the same sign, and we have the inequalities (we can put $k=1$ without loss of generality)

$$\sum_{\nu=2}^{m}(a_{\nu s}+b_{\nu s})^2+4a_{1s}b_{1s}<0 \tag{2.7}$$

then the zero solution of system (2.1) is Lyapunov-stable.

In the case when there are inner resonances, the sufficient condition for the zero solution of system (2.1) to be stable is that both inequality (2.7) and an inequality similar to 2) of Theorem 1 hold.

3. When studying the stability of the rotations of a celtic stones /11/, the case $m=1$, $n=2$ (system (3.10)) was considered, though the coefficients b_{1s} have different signs. Consequently, the stability theorems, whether there are no inner resonances /10/ or there are (Theorem 1, Note 1°), are not applicable for the case taken in /11/, though the K-singularity remains.

Assuming that K-singularity is present, we consider the case when $m=1$, $n\geqslant 2$, though the coefficients b_{1s} are not all of the same sign, and there are not inner resonances. We rewrite system (2.5), indicating explicitly the form of the cubic terms and putting $M=n+1$ (the equations for $\theta_s^{\cdot}$ are omitted):

$$\xi_1^{\cdot}=\sum_{s=1}^{n}\rho_s^2\Big[a_{1s}+\alpha_{1s}\xi_1+\sum_{l=2}^{n-1}H_s^{(l)}(\xi_1,\rho)\Big]+H_1(\xi_1,\rho,\theta,t) \tag{3.1}$$

$$\rho_s^{\cdot}=\rho_s\Big[b_{1s}\xi_1+\beta_{s0}\xi_1^2+\sum_{\nu=1}^{n}\beta_{s\nu}\rho_\nu^2+\sum_{l=3}^{n}P_s^{(l)}(\xi_1,\rho)\Big]+P_s(\xi_1,\rho,\theta,t)$$

where the forms $H_s^{(l)}$, $P_s^{(l)}$ of order l are not in general the same as in (2.5).

Theorem 3. In the K-singular case with $m=1$, $n\geqslant 2$, when there are not inner resonances, suppose we have for all s the inequalities: 1) $a_{1s}b_{1s}<0$; 2) $b_{1s}+b_{1\nu}\neq 0$, $\nu=1,\ldots,n$; 3) $\beta_{s\nu}<0$, $\nu=1,\ldots,n$; 4) $\alpha_{1s}-a_{1s}\beta_{s0}/b_{1s}<0$.

Then, the zero solution of system (3.1) is Lyapunov-stable.

Proof. Let us explain the structure of the leading non-linear terms. Since we have K-singularity, and as noted above, the functions $F_0^{(2)}$, F_1 contain no terms linear in η, $\bar\eta$, z in system (2.1), the functions H_1 and P_s can be written as

$$H_1 = \sum_{\nu=1}^{n} \left[\rho_\nu^2 H_{\nu 1}^{(n)}(\rho, \theta, t) + \rho_\nu^2 \xi_1 H_{\nu 2}^{(n-1)}(\rho, \theta, t) + \sum_{j=1}^{n} \rho_\nu \rho_j (\xi_1^2 H_{\nu j 3}^{(n-2)}(\xi_1, \rho, \theta, t) + H_{\nu j 4}^{(n)}(\xi_1, \theta, t))\right] \tag{3.2}$$

$$P_s = \sum_{\nu=1}^{n} [\rho_\nu^2 (P_{s\nu 1}^{(n)}(\rho, \theta, t) + \xi_1 P_{s\nu 2}^{(n-1)}(\rho, \theta, t) + \rho_\nu (\xi_1^2 P_{s\nu 3}^{(n-1)}(\xi_1, \rho, \theta, t) + P_{s\nu 4}^{(n+1)}(\xi_1, \theta, t))]$$

where $H_{\nu 1}^{(n)}, \ldots, H_{\nu j4}^{(n)}, P_{s\nu 1}^{(n)}, \ldots, P_{s\nu 4}^{(n+1)}$ are functions periodic in θ and t, and their order of smallness with respect to ξ_1 and ρ is not less than the corresponding superscripts.

Since the normalization /7, 8/ in system (3.1) is carried out up to and including terms of the $(n+1)$-th order, we can write all the forms $P_s^{(l)}$ as

$$P_s^{(l)} = D_{sl} \xi_1^l + \sum_{\nu=1}^{n} \rho_\nu^2 D_{s\nu}^{(l-2)}(\xi_1, \rho) \tag{3.3}$$

where D_{sl} are real constants, and $D_{s\nu}^{(l-2)}$ are forms of order $(l-2)$ in the variables ξ_1, ρ.

We consider the function

$$V = \frac{\xi_1^2}{2} + \sum_{s=1}^{n} \left(-\frac{a_{1s}}{b_{1s}}\right) \frac{\rho_s^2}{2} + \sum_{s,\nu=1}^{n} K_{s\nu} \xi_1 \rho_s \rho_\nu$$

where $K_{s\nu}$ are real constants. By condition 1) of the theorem, given any fixed $K_{s\nu}$ and sufficiently small ξ_1 and ρ, the function V is positive definite.

Using system (3.1) and the structure of relations (3.2), the derivative $V^\cdot$ is

$$V^\cdot = \sum_{s=1}^{n} \rho_s^2 \left[\left(\alpha_{1s} - a_{1s}\frac{\beta_{s0}}{b_{1s}} + 2K_{ss} b_{1s}\right) \xi_1^2 + a_{1s} \sum_{\nu, j=1}^{n} K_{\nu j} \rho_\nu \rho_j + \sum_{\nu=1}^{n} \beta_{s\nu} \left(-\frac{a_{1\nu}}{b_{1\nu}}\right) \rho_\nu^2 + \cdots\right] + \sum_{\substack{s,\nu=1 \\ s \neq \nu}}^{n} \xi_1^2 \rho_s \rho_\nu [K_{s\nu}(b_{1s} + b_{1\nu}) + \cdots]$$

Here, the dots denote terms of higher order of smallness than those written.

For the derivative $V^\cdot$ to be of fixed sign (such that $VV^\cdot \leqslant 0$), the sufficient conditions are

A) $\alpha_{1s} - a_{1s}\dfrac{\beta_{s0}}{b_{1s}} + 2K_{ss} b_{1s} < 0$ for all s;

B) $V_s(\rho) = \left[a_{1s} \sum_{\nu, j=1}^{n} K_{\nu j} \rho_\nu \rho_j + \sum_{\nu=1}^{n} \beta_{\nu s} \left(-\frac{a_{1\nu}}{b_{1\nu}}\right) \rho_\nu^2\right]$

are negative definite functions of ρ for all s;

C) $K_{s\nu}(b_{1s} + b_{1\nu}) < 0$ for all $\nu = 1, \ldots, n$ and $s\ (\nu \neq s)$

We put all the $K_{ss} = 0$, and choose the $K_{s\nu}$ such that conditions c) are satisfied and are so small that, for all s, the functions $V_s(\rho)$ are negative definite (this is possible by virtue of conditions 1), 2), 3) of the theorem). Conditions A) then hold by virtue of condition 4). Hence V is a Lyapunov function, where $VV^\cdot \leqslant 0$.

Note. Theorem 3 can be strengthened if the necessary and sufficient conditions /12/ for fixed sign in a cone are applied to functions $V_s(\rho)$; but we refrain from doing this because of the unwieldy working involved.

4. Consider the instability. It was shown in /13/ that, if there are no inner resonances of the third order, and there exists, with $m = 1$, at least one pair of numbers a_{1s}, b_{1s}, satisfying the inequality $a_{1s} b_{1s} > 0$, then the zero solution of system (2.5) is unstable. This assertion obviously remains true in the K-singular case for any $n \geqslant 1$.

Theorem 4. Let $m = 2$, let there be no third-order inner resonances, and let us have K-singularity for $n \geqslant 1$. Then, if we have at least one of the inequalities

$$a_{1s}b_{1s} + a_{2s}b_{2s} > 0 \tag{4.1}$$

the zero solution of system (2.5) is unstable.

Proof. Assume for clarity that (4.1) holds with $s = 1$. Using the usual arguments (see e.g., /8/), we can discard the equation for $\rho_s\theta_s^{\cdot}$ in system (2.5) and consider the truncated system /8/

$$\xi_1^{\cdot} = \sum_{\nu=1}^{n} a_{1\nu}\rho_\nu^2, \quad \xi_2^{\cdot} = \sum_{\nu=1}^{n} a_{2\nu}\rho_\nu^2, \quad \rho_s^{\cdot} = (b_{1s}\xi_1 + b_{2s}\xi_2)\,\rho_s \tag{4.2}$$

In the same way as in Kamenkov's instability theorem /6/, we form the following functions by noting the form of the right-hand sides of system (4.2) (it can be assumed without loss of generality that $a_{11} \neq 0$):

$$T_0 = \sum_{\nu=1}^{n} \rho_\nu^2 (a_{1\nu}\xi_2 - a_{2\nu}\xi_1), \quad T_s = \rho_s \Big[\sum_{\nu=1}^{n} a_{1\nu}\rho_\nu^2 - b_{1s}\xi_1^2 - b_{2s}\xi_1\xi_2\Big], \tag{4.3}$$
$$R = \sum_{s=1}^{n} \rho_s^2 \left[(a_{1s} + b_{1s})\,\xi_1 + (a_{2s} + b_{2s})\,\xi_2\right]$$

Notice that, when $\xi_1 \neq 0$, expressions $\xi_2 = a_{21}\xi_1/a_{11}$, $\rho_1^2 = (a_{11}b_{11} + a_{21}b_{21})(\xi_1/a_{11})^2$, $\rho_\nu = 0$ ($\nu = 2, \ldots, n$) are non-trivial solutions of the system of algebraic equations

$$T_0 = 0, \quad T_s = 0 \quad (s = 1, \ldots, n)$$

while the function R on this solution takes the form $R = \rho_1^2\,(a_{11}b_{11} + a_{21}b_{21})\xi_1/a_{11}$. Consequently, if we assume that $\xi_1/a_{11} > 0$, all the conditions of Kamenkov's instability theorem /6/ are formally satisfied: all the ρ_s are of the same sign, i.e., they cannot be arbitrary, as is implicitly assumed in this theorem. We therefore pass to new variables w_1, w_2, u_s, given by the relations

$$w_1 = \xi_1/a_{11}, \; w_2 = \xi_2 - a_{21}w_1, \; u_1 = \rho_1 - (a_{11}b_{11} + a_{21}b_{21})^{1/2}w_1,$$
$$u_\nu = \rho_\nu - w_2 \quad (\nu = 2, \ldots, n)$$

This change of variables obviously does not alter the stability problem. We rewrite system (4.2) in the new variables and, for the new system, form functions similar to (4.3), and thus arrange for all the conditions of Kamenkov's instability theorem to be satisfied, which can be done simply by taking $w_1 = \xi_1/a_{11} > 0$. Hence our theorem follows.

We will now state a trivial extension of Theorem 4.

Theorem 5. Let $m > 2$, let there be no third-order inner resonances, and let us have K-singularity for $n \geqslant 1$. Then, if at least one of the conditions

$$\sum_{k=1}^{m} a_{ks}b_{ks} > 0$$

holds, the zero solution of system (2.5) is unstable.

5. In the model of a celtic stone /14/ it is assumed that the principal **radii of curvature** r_1 and r_2 are different. Retaining the notation of /14/, we consider the case when $r_1 = r_2$ and we have the inequalities

$$\Delta_0 \equiv D^2 - dd_1 \neq 0, \quad A_0 \equiv 2\omega^2 + [\omega^2 (dd_1 - \alpha\alpha_1) + \delta\,(d + d_1)]\Delta_0^{-1} >$$
$$0, \quad B_0 \equiv \omega^4 + [\omega^4 (dd_1 - \alpha\alpha_1) - \delta\,(\delta + \omega^2 (\alpha + \alpha_1))]\Delta_0^{-1} > 0,$$
$$A_0^2 - 4B_0 > 0$$

The matrix of the linear approximation of the equations of disturbed motion (rotation of the celtic stone about a vertical axis on an absolutely rough plane) then has two zeros, and a pair of pure imaginary distinct, eigenvalues $\pm i\omega\varkappa_1$, $\pm i\omega\varkappa_2$, where

$$\varkappa_{1,2} = ((A_0 \pm (A_0^2 - 4B_0)^{1/2})/2\omega^2)^{1/2}$$

and we have K-singularity. Up to and including quadratic terms, the equations of disturbed motion are

$$p^{\cdot} = A_{11}p - A_{12}q + A_{13}\gamma_1 - A_{14}\gamma_2 + (A_{11}/\omega)p\beta_1 - (A_{12}/\omega)q\beta_1 + \ldots, \tag{5.1}$$
$$q^{\cdot} = A_{21}p - A_{11}q + A_{23}\gamma_1 - A_{13}\gamma_2 + (A_{21}/\omega)p\beta_1 - (A_{11}/\omega)q\beta_1 + \ldots$$
$$\gamma_1^{\cdot} = -q + \omega\gamma_2 - q\beta_2 + \beta_1\gamma_2, \quad \gamma_2^{\cdot} = p - \omega\gamma_1 + p\beta_2 - \beta_1\gamma_1$$
$$\beta_1^{\cdot} = (D/C)p^2 + ((A - B)/C)pq - (D/C)q^2 + A_{11}Gp\gamma_1 + (E/C +$$
$$A_{21}G)p\gamma_2 - (E/C + A_{12}G)q\gamma_1 - A_{11}Gq\gamma_2 + A_{13}G\gamma_1^2 + (A_{23} - A_{14})G\gamma_1\gamma_2 -$$
$$A_{13}G\gamma_2^2 + \ldots, \quad \beta_2^{\cdot} = q\gamma_1 - p\gamma_2$$

$$A_{11} = D\omega(\alpha + d)\Delta_0^{-1}, \quad A_{12} = \omega(D^2 + \alpha d_1)\Delta_0^{-1}$$
$$A_{21} = \omega(D^2 + \alpha_1 d)\Delta_0^{-1}, \quad A_{13} = D\delta\Delta_0^{-1}, \quad A_{14} = d_1\delta\Delta_0^{-1}$$
$$A_{23} = d\delta\Delta_0^{-1}, \quad E = Mr_1\omega(h - r_1), \quad G = Mhr_1/C.$$

The remaining notation is the same as in /14/.

After the change of variables $z_1 = p + iq, \bar{z}_1 = p - iq, z_2 = \gamma_1 + i\gamma_2, \bar{z}_2 = \gamma_1 - i\gamma_2$ and subsequent normalization /8/, Eqs.(5.1) become

$$\beta_1^{\cdot} = a_{11}\rho_1^2 + a_{12}\rho_2^2 + \ldots, \quad \beta_2^{\cdot} = 0 + \ldots$$
$$\rho_1^{\cdot} = b_{11}\beta_1\rho_1 + \ldots, \quad \rho_2^{\cdot} = b_{12}\beta_1\rho_2 + \ldots$$

where the dots denote small third-order terms in $\beta_1, \beta_2, \rho_1, \rho_2$ (the equations for the angular variables are omitted), while the coefficients of this system are given by

$$a_{1s} = {}^1/_2\omega^2(1 - \varkappa_s^2)[2D(K_s + \bar{K}_s) + i(A - B)(K_s - \bar{K}_s)] +$$
$$\omega Mhr_1[b_0\bar{K}_s + \bar{b}_0K_s] + Mhr_1[d_0\bar{K}_s + \bar{d}_0\bar{K}_s], \quad s = 1, 2$$
$$b_{11} = \omega^{-1}S^{-1}[4\varkappa_1\varkappa_2(\omega\,\mathrm{Im}(K_1\bar{K}_2) - \mathrm{Re}(\bar{K}_1K_2\bar{a}_0)) + 2\varkappa_1(1 - \varkappa_2 -$$
$$(1 + \varkappa_2)K_2\bar{K}_2)\mathrm{Re}(b_0K_1) + 2\varkappa_2\,\mathrm{Re}(\bar{b}_0\bar{K}_2(1 + \varkappa_1) - b_0K_2K_1\bar{K}_1(1 - \varkappa_1))]$$
$$b_{12} = \omega^{-1}S^{-1}[4\varkappa_1\varkappa_2(\omega\,\mathrm{Im}(K_1\bar{K}_2) - \mathrm{Re}(\bar{K}_1K_2a_0)) - 2\varkappa_2(1 + \varkappa_1 -$$
$$(1 - \varkappa_1)K_1\bar{K}_1)\mathrm{Re}(b_0K_2) + 2\varkappa_1\,\mathrm{Re}(-\bar{b}_0\bar{K}_1(1 - \varkappa_2) + b_0K_1K_2\bar{K}_2(1 + \varkappa_2))]$$
$$K_1 - \frac{\bar{b}_0\omega(1 + \varkappa_1) + d_0}{(\bar{a}_0 \pm i\omega\varkappa_1)\omega(1 - \varkappa_1) + c_0}, \quad K_2 - \frac{\bar{b}_0\omega(1 - \varkappa_2) + \bar{d}_0}{(\bar{a}_0 + i\omega\varkappa_2)\omega(1 + \varkappa_2) + c_0}$$
$$S = (1 + K_1\bar{K}_1K_2\bar{K}_2)(\varkappa_1 + \varkappa_2)^2 - (\varkappa_1 - \varkappa_2)^2(K_1\bar{K}_1 + K_2\bar{K}_2) -$$
$$4\varkappa_1\varkappa_2(K_1\bar{K}_2 + \bar{K}_1K_2), \quad a_0 = i(A_{21} + A_{12})/2, \quad c_0 = i(A_{23} + A_{14})/2$$
$$b_0 = A_{11} + i(A_{21} - A_{12})/2, \quad d_0 = A_{13} + i(A_{23} - A_{14})/2$$

Notice that, when ω is replaced by $-\omega$, the coefficients a_{11}, a_{12}, b_{11} and b_{12} remain unchanged. Applying Theorem 1, we see that, with $a_{11}b_{11} < 0, a_{12}b_{12} < 0, b_{11}b_{12} > 0$, the zero solution of system (5.1) is stable, while it is unstable if we have one or both of the inequalities $a_{11}b_{11} > 0, \ a_{12}b_{12} > 0$ (Theorem 4).

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